

# Momentum-Space Topology of Standard Model

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*The momentum-space topological invariants, which characterize the ground state of the Standard Model, are continuous functions of two parameters, generated by the hypercharge and by the weak charge. These invariants provide the absence of the mass of the elementary fermionic particles in the symmetric phase above the electroweak transition (the mass protection). All the invariants become zero in the broken symmetry phase, as a result all the elementary fermions become massive. Relation of the momentum-space invariants to chiral anomaly is also discussed.*

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## 1. INTRODUCTION

There is an important class of the 3+1 dimensional Fermi systems. The systems which belong to this class are characterized by the Fermi points instead of the Fermi surfaces. Among the representatives of this class are the vacuum of Standard Model of electroweak and strong interactions in high energy physics and the superfluid  $^3\text{He-A}$  in condensed matter physics.<sup>1</sup> Fermi point is the point in the momentum space  $\mathbf{p}$  where the (quasi)particle energy is zero. In particle physics the energy spectrum  $E(\mathbf{p}) = cp$  is characteristic of the massless neutrino (or any other chiral lepton or quark in the Standard Model) with  $c$  being the speed of light. The energy of the chiral neutrino is zero at point  $\mathbf{p} = 0$  in the 3D momentum space. In condensed matter systems the points with zero energy have been realized first in superfluid  $^3\text{He-A}$ , where the superfluid gap has two point nodes. Another example of the Fermi point in condensed matter system has been discussed for gapless

semiconductors.<sup>2</sup>

The massless (gapless) character of the fermionic spectrum in the system with the Fermi point is protected by the topological invariant of the ground state, which is expressed as the integral over the Green's function in the 4D momentum-frequency space:

$$N = \text{tr } \mathcal{N} \quad , \quad \mathcal{N} = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma} \int_{\sigma} dS^{\gamma} \mathcal{G} \partial_{p_{\mu}} \mathcal{G}^{-1} \mathcal{G} \partial_{p_{\nu}} \mathcal{G}^{-1} \mathcal{G} \partial_{p_{\lambda}} \mathcal{G}^{-1} . \quad (1)$$

The integral here is over the surface  $\sigma$  embracing the point  $\mathbf{p} = 0, p_0 = 0$ . in the 4D momentum space  $\mathbf{p}, p_0$ , and  $p_0$  is the energy (frequency) along the imaginary axis;  $\text{tr}$  is the trace over the fermionic indices.

As an example let us consider the chiral spin-1/2 particle. In the limit of the noninteracting particles, the Green's function matrix considered on the imaginary frequency axis,  $z = ip_0$ , is

$$\mathcal{G} = (ip_0 - \mathcal{H})^{-1} \quad , \quad \mathcal{H} = \pm c \vec{\sigma} \cdot \mathbf{p} . \quad (2)$$

Here  $\vec{\sigma}$  are  $2 \times 2$  Pauli spin matrices, so that  $\text{tr}$  in Eq.(1) is the trace over the spin indices; the sign  $+$  is for a right-handed particle and  $-$  for a left-handed one; the spin of the particle is oriented along or opposite to its momentum, respectively. The Green's function has a singularity at the point  $\mathbf{p} = 0, p_0 = 0$ , and such singularity cannot be removed because of the momentum-space topology. Substituting this Green's function into the topological invariant, Eq.(1), one obtains that this invariant is nonzero: it is  $N = 1$  for the righthanded particle and  $N = -1$  for the lefthanded one.

What is important here that the Eq.(1), being the topological invariant, does not change under any (but not very large) perturbations. This means that even if the interaction between the particles is introduced and the Green's functions changes drastically, the result remains the same:  $N = 1$  for the righthanded particle and  $N = -1$  for the lefthanded one. The singularity of the Green's function remains, which means that the quasiparticle spectrum remain gapless: fermions are massless even in the presence of interaction. The nonzero value of the momentum-space topological invariant thus provides the mass protection for fermions. This mass protection mechanism based on the topological properties in the momentum space is ideologically different from that based on gauge invariance arguments.<sup>3</sup>

Now let us consider what happens if there are several fermionic species. In this case the trace operation is also over all the fermions. If the fermionic system has an equal number of lefthanded and righthanded fermions, then the topological invariant in Eq.(1) is  $N = 0$ , and the above mass protection mechanism does not work. This situation occurs in the planar phase of superfluid  $^3\text{He}$  (for planar phase see<sup>4</sup>) and the Standard Model. If the righthanded

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neutrino is present, as follows from the Kamiokande experiments, then each generation contains 8 lefthanded and 8 righthanded fermions.

Here we show that, as in the case of the planar state of superfluid  $^3\text{He}$ , there are modified topological invariants which provide the mass protection even for equal number of left and right fermions.

### 2. Generating Function for Invariants

One can introduce these invariants in the following way:

$$N(F) = \text{tr} [\mathcal{N}\mathcal{F}] \ , \quad (3)$$

where  $\mathcal{F}$  is any matrix function of any operator which commutes with the Green's function  $\mathcal{G}$ . For example, in the Standard Model above the electroweak transition there is a  $U(1)_Y$  symmetry generated by the hypercharge  $Y$ . Since the hypercharge matrix  $\mathcal{Y}$  commutes with the Green's function, the matrix  $\mathcal{F}$  can be any power of the hypercharge,  $\mathcal{F} = \mathcal{Y}^n$ . One can easily verify that the perturbations of the Green's function, which conserve the  $U(1)_Y$  symmetry, do not change the integral  $N(Y^n)$ . In the planar phase of  $^3\text{He}$  the corresponding  $U(1)$  symmetry is combined rotations in spin and orbital space with generator  $J_3 = S_3 + L_3$ .

Let us introduce the generating function for all the topological invariants containing powers of the hypercharge

$$N(\theta_Y) = \text{tr} \left[ e^{i\theta_Y \mathcal{Y}} \mathcal{N} \right] \ . \quad (4)$$

All the powers  $\text{tr} [\mathcal{Y}^n \mathcal{N}]$  can be obtained by differentiating over the phase angle parameter  $\theta_Y$ . Since the above parameter-dependent invariant is robust to interactions between the fermions, it can be calculated for the noninteracting particles. In the latter case the matrix  $\mathcal{N}$  is diagonal with the eigenvalues  $+1$  and  $-1$  for right and left fermions correspondingly. The trace of this matrix  $\mathcal{N}$  over given irreducible fermionic representation of the gauge group is (with minus sign) the symbol  $N_{(y/2, \underline{a}, I_W)}$  introduced by Froggatt and Nielsen in Ref.<sup>3</sup> In their notations  $y/2 (= Y)$ ,  $\underline{a}$ , and  $I_W$  denote hypercharge, colour representation and the weak isospin correspondingly.

For the Standard model with right neutrino included one has

$$\text{tr} \left[ e^{i\theta_Y \mathcal{Y}} \mathcal{N} \right] = -\frac{1 - \cos \theta_Y/2}{1 + \cos \theta_Y/2} \text{tr} \left[ e^{i\theta_Y \mathcal{Y}} \right] \ . \quad (5)$$

Here we used the values of the hypercharge:  $Y = 1/6$  for left quarks;  $Y = 2/3$  and  $Y = -1/3$  for right quarks;  $Y = -1/2$  for left leptons;  $Y = -1$  for right electron, muon,  $\tau$  lepton, etc.; and  $Y = 0$  for right neutrinos.

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In addition to the hypercharge the weak charge is also conserved in the Standard model above the electroweak transition. The generating function for the topological invariants which contain the powers of both the hypercharge  $Y$  and the weak charge  $W_3$  has the form

$$\text{tr} \left[ e^{i\theta_W \mathcal{W}_3} e^{i\theta_Y \mathcal{Y}} \mathcal{N} \right] = \frac{\cos \theta_Y/2 - \cos \theta_W/2}{\cos \theta_Y/2 + \cos \theta_W/2} \text{tr} \left[ e^{i\theta_W \mathcal{W}_3} e^{i\theta_Y \mathcal{Y}} \right]. \quad (6)$$

Finally one should add the powers of the generators  $\mathcal{T}_{C3}$  and  $\mathcal{T}_{C8}$  of the  $SU(3)$  colour group. They, however, do not change the form of the generating function in Eq.(6):

$$\frac{\text{tr} \left[ e^{i\theta_W \mathcal{W}_3} e^{i\theta_Y \mathcal{Y}} e^{i\theta_{C3} \mathcal{T}_{C3}} e^{i\theta_{C8} \mathcal{T}_{C8}} \mathcal{N} \right]}{\text{tr} \left[ e^{i\theta_W \mathcal{W}_3} e^{i\theta_Y \mathcal{Y}} e^{i\theta_{C3} \mathcal{T}_{C3}} e^{i\theta_{C8} \mathcal{T}_{C8}} \right]} = \frac{\cos \theta_Y/2 - \cos \theta_W/2}{\cos \theta_Y/2 + \cos \theta_W/2}. \quad (7)$$

One could add the powers of the generators of the groups which correspond to conservation of the baryonic and leptonic charges. But this also does not change the above result: the normalized topological invariants depend only on two angle-phase parameters  $\theta_W$  and  $\theta_Y$ .

It is the nonzero function of the parameters determining the momentum-space invariants,

$$\frac{\cos \theta_Y/2 - \cos \theta_W/2}{\cos \theta_Y/2 + \cos \theta_W/2}, \quad (8)$$

that provides the mass protection for the Standard Model.

### 3. Relation to Axial Anomaly

The momentum-space topological invariants determine the axial anomaly in fermionic systems. In the 2+1 condensed matter, the  $\theta$ -factor in front of the Chern-Simons term is proportional to the invariant Eq.(1), where the integration region  $\sigma$  is the whole 2+1 momentum space. For Quantum Hall Effect in electronic systems this relation was established in.<sup>5</sup> For  $^3\text{He-A}$  films, the Chern-Simons action for unit vector  $\hat{d}$  is<sup>6</sup>

$$S_\theta = \frac{\hbar\theta}{32\pi^2} \int d^2x dt e^{\mu\nu\lambda} A_\mu F_{\nu\lambda}, \quad F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu = \hat{d} \cdot \partial_\nu (\hat{d} \times \partial_\lambda \hat{d}), \quad (9)$$

with

$$\theta = \frac{\pi}{2} \text{tr} \mathcal{N}. \quad (10)$$

Here, in addition to spin indices, the matrix  $\mathcal{N}$  contains the indices of the transverse levels, which come from quantization of motion along the normal to the film. The transverse levels play the role of different families of fermions.

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The action in Eq.(9) represents the product of the topological invariants in real and momentum spaces. For more general 2+1 condensed matter systems with different types of momentum-space invariants, see.<sup>7</sup>

The Eq.(7), which generates the momentum-space invariants for the Standard Model, is also related to anomalies. The Wess-Zumino action, which describes the anomaly, can be also represented in terms of the product of the real space and momentum space invariants. In particular the axial anomaly term generated by the hypermagnetic field  $A_\mu^Y$  is

$$S_{\text{WZ}}^Y = \frac{\hbar}{96\pi^2} \int d^5x \, e^{\mu\nu\lambda\alpha\beta} A_\mu^Y F_{\nu\lambda}^Y F_{\alpha\beta}^Y \, \text{tr} [\mathcal{Y}^3 \mathcal{N}] . \quad (11)$$

In the Standard model such action is zero because of the anomaly cancellation,  $\text{tr} [\mathcal{Y}^3 \mathcal{N}] = 0$ , which follows from the Eq.(7). The other Wess-Zumino terms are also zero, again due to anomaly cancellation expressions which follow from the Eq.(7):

$$\text{tr} [\mathcal{Y} \mathcal{N}] = \text{tr} [\mathcal{Y}^3 \mathcal{N}] = \text{tr} [\mathcal{Y} \mathcal{W}_3^2 \mathcal{N}] = \text{tr} [\mathcal{Y}^2 \mathcal{W}_3 \mathcal{N}] = \dots = 0 . \quad (12)$$

## 4. Masses via Symmetry Breaking

The mass of the fermions can appear only if the  $U(1)_Y \times SU(2)$  symmetry is violated. Then the hypercharge and weak charge are no more conserved quantity. The symmetry breaking which removes the mass protection can be for example the complete symmetry breaking,  $U(1)_Y \times SU(2) \rightarrow 1$ . In this case the only relevant topological invariant which is left below the transition is Eq.(1), but it is zero for the discussed model. Thus there is no mass protection below such transition: all the fermions must have the mass.

There is another possibility, which also removes the mass protection. It is the partial symmetry breaking  $U(1)_Y \times SU(2) \rightarrow U(1)_Q$ , where the generator  $Q$  can be either  $Q = \pm(Y - W_3)$  or  $Q = \pm(Y + W_3)$ . As follows from Eq.(6), the generating function for the topological invariants which contain the powers of  $Q$  is zero. For example, for  $Q = Y + W_3$ , which corresponds to the electric charge, one has

$$\text{tr} [e^{i\theta_Q Q} \mathcal{N}] = \text{tr} [e^{i\theta_Q \mathcal{Y}} e^{i\theta_Q \mathcal{W}_3} \mathcal{N}] = 0 , \quad (13)$$

since this is the Eq.(6) with  $\theta_Y = \theta_W$ .

The nature had chosen the electric charge to be  $Q = Y + W_3$  and thus each elementary fermion in our world has a mass.

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## 5. Discussion

What is the reason for such a choice? Why the nature had not chosen the more natural symmetry breaking, such as  $U(1)_Y \times SU(2) \rightarrow U(1)_Y$ ,  $U(1)_Y \times SU(2) \rightarrow SU(2)$  or  $U(1)_Y \times SU(2) \rightarrow U(1)_Y \times U(1)_{W_3}$ ? Or why it did not consider the symmetry breaking  $U(1)_Y \times SU(2) \rightarrow U(1)_Q$ , with  $Q = nY + mW_3$  and  $n \neq \pm m$ ? Probably this is because in all the above cases the mass protection remains even below the transition, and all fermions remain massless. This can shed light on the origin of the electroweak transition. Maybe the elimination of the mass protection is the only purpose of the transition. This is similar to the Peierls transition in condensed matter: the formation of mass (gap) is not the consequence but the cause of the transition. It might be energetically favourable to have masses of quasiparticles, since this leads to decrease of the energy of the Fermi sea. Formation of the condensate of top quarks, which generates the heavy mass of the top quark, could be a relevant scenario (see review<sup>8</sup>).

Another question, why the hypercharges are so nicely organized that the nullification of all the invariants  $\text{tr } [Q^n \mathcal{N}]$  is realized? The possible answer is that the group  $SU(3) \times SU(2) \times U(1)_Y$  is embedded in the higher symmetry group, which contains the left-right symmetric operator  $Q$ . For example, it can be the left-right symmetric model of Pati-Salam type<sup>9</sup> with symmetry group  $SU(4) \times SU(2)_L \times SU(2)_R$ . It organizes 16 fermions of each generation into the left and right baryon-lepton octets:

$$SU(4) \quad \begin{pmatrix} SU(2)_L \\ \begin{pmatrix} u_L & d_L \\ u_L & d_L \\ u_L & d_L \\ \nu_L & e_L \end{pmatrix} \end{pmatrix} \quad \begin{pmatrix} SU(2)_R \\ \begin{pmatrix} u_R & d_R \\ u_R & d_R \\ u_R & d_R \\ \nu_R & e_R \end{pmatrix} \end{pmatrix} \quad (14)$$

In addition to the  $SU(3)$  charges there are the right and left weak charges  $W_{R3}$  and  $W_{L3}$ , and the difference between the baryonic and leptonic numbers  $B - L$ , which comes from the extension of the colour group  $SU(3)$  to  $SU(4)$ :

<i>Fermion</i>	$W_{L3}$	$W_{R3}$	$B - L$
$u_L(3)$	$+\frac{1}{2}$	0	$\frac{1}{3}$
$d_L(3)$	$-\frac{1}{2}$	0	$\frac{1}{3}$
$u_R(3)$	0	$+\frac{1}{2}$	$\frac{1}{3}$
$d_R(3)$	0	$-\frac{1}{2}$	$\frac{1}{3}$
$e_L$	$-\frac{1}{2}$	0	-1
$\nu_L$	$+\frac{1}{2}$	0	-1
$e_R$	0	$-\frac{1}{2}$	-1
$\nu_R$	0	$+\frac{1}{2}$	-1

(15)

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The generating function relevant for this group is

$$\text{tr} \left[ e^{i\theta_R \mathcal{W}_3^R} e^{i\theta_L \mathcal{W}_3^L} e^{i\theta_{BL}(\mathcal{B}-\mathcal{L})} \mathcal{N} \right] = 2 \left( \cos \frac{\theta_R}{2} - \cos \frac{\theta_L}{2} \right) \left( 3e^{i\theta_{BL}/3} + e^{-i\theta_{BL}} \right). \quad (16)$$

The 16 fermions of one generation can be represented as the product  $Cw$  of 4 bosons and 4 fermions.<sup>10</sup> This scheme is similar to the slave-boson approach in condensed matter, where the particle is considered as a product of the spinon and holon. Spinons are fermions which carry spin, while holons are "slave"-bosons which carry electric charge.<sup>11</sup> In the Terazawa scheme the "holons"  $C$  form the  $SU(4)$  quartet of spin-0  $SU(2)$ -singlet particles which carry baryonic and leptonic charges, their  $B-L$  charges of the  $SU(4)$  group are  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$ . The "spinons" are spin- $\frac{1}{2}$  particles  $w$ , which are  $SU(4)$  singlets and  $SU(2)$ -isodoublets; they carry spin and isospin.

$$\begin{pmatrix} u_L & d_L & u_R & d_R \\ u_L & d_L & u_R & d_R \\ u_L & d_L & u_R & d_R \\ \nu_L & e_L & \nu_R & e_R \end{pmatrix} = \begin{pmatrix} C_{1/3} \\ C_{1/3} \\ C_{1/3} \\ C_{-1} \end{pmatrix} \times \begin{pmatrix} w_L^{+1/2} & w_L^{-1/2} & w_R^{+1/2} & w_R^{-1/2} \end{pmatrix} \quad (17)$$

Here  $\pm 1/2$  means the charge  $W_{L3}$  for the left spinons and  $W_{R3}$  for the right spinons, which coincides with the electric charge of spinons:  $Q = \frac{1}{2}(B-L) + W_{L3} + W_{R3} = W_{L3} + W_{R3}$ . In Terazawa notations  $w_1 = (w_L^{+1/2}, w_R^{+1/2})$  forms the doublet of spinons with  $Q = +1/2$  and  $w_2 = (w_L^{-1/2}, w_R^{-1/2})$  - with  $Q = -1/2$ . These 4 spinons, 2 left and 2 right, transform under  $SU(2)_L \times SU(2)_R$  symmetry group. The generating function for the momentum space topological invariants for spinons is  $\text{tr} \left[ e^{i\theta_R \mathcal{W}_3^R} e^{i\theta_L \mathcal{W}_3^L} \mathcal{N} \right] = 2(\cos \frac{\theta_R}{2} - \cos \frac{\theta_L}{2})$ . The factorization in the Eq.(16) thus reflects the factorization in Eq.(17).

The electric charge  $Q = \frac{1}{2}(B-L) + W_{L3} + W_{R3}$  in the above scheme is left-right symmetric. That is why, if only the electric charge is conserved in the final broken symmetry state, the only relevant topological invariant  $\text{tr} \left[ e^{i\theta_Q \mathcal{Q}} \mathcal{N} \right]$  becomes zero and the Weyl fermions can be paired into Dirac fermions. This fact does not depend on the definition of the hypercharge, which appears at the intermediate stage where the symmetry is  $SU(3) \times SU(2) \times U(1)_Y$ . It also does not depend much on the definition of the electric charge  $Q$  itself: the only condition for the nullification of the topological invariant is the symmetry (or antisymmetry) of  $Q$  with respect to the parity transformation.

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